Overview of the standards
Algebra standards and structures
Sample tasks

Common Core State Standards for Mathematics

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Algebra Forum, San Jose, 18 October 2010
Structure of the standards

- Standards for Mathematical Practice
Structure of the standards

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  - Carry across all grade levels
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  - Describe habits of mind of a mathematically expert student
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  - Organized into domains that progress over several grades
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  - K–8 standards presented by grade level
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  - Grade introductions give 2–4 focal points at each grade level
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- Standards for Mathematical Content
  - K–8 standards presented by grade level
  - Organized into domains that progress over several grades
  - Grade introductions give 2–4 focal points at each grade level
  - High school standards presented by conceptual theme (Number & Quantity, Algebra, Functions, Modeling, Geometry, Statistics & Probability)
Standards for mathematical practice

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning
Arrangement of content standards

**Number and Operations in Base Ten**

**3.NBT**

*Use place value understanding and properties of operations to perform multi-digit arithmetic.*

1. Use place value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

- **Content standards** define what students should understand and be able to do
- **Clusters** are groups of related standards
- **Domains** are larger groups that progress across grades
# Domains K–8

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<tr>
<th>Counting &amp; Cardinality</th>
<th>Ratios &amp; Proportional Relationships</th>
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<tr>
<td>Operations and Algebraic Thinking</td>
<td>The Number System</td>
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<tr>
<td>Number and Operations in Base Ten</td>
<td>Expressions and Equations</td>
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<td>Fractions</td>
<td>Functions</td>
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<table>
<thead>
<tr>
<th>K</th>
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Common Core State Standards for Mathematics
High school “super-domains”

- Number and Quantity
- Algebra
- Functions
- Modeling
- Geometry
- Statistics and Probability
- (+) standards indicate material beyond the college and career ready threshold; can be in courses required for all students
- ★ indicates modeling standards, distributed throughout
Flows leading to algebra

Operations and Algebraic Thinking \(\rightarrow\) Expressions and Equations

Number and Operations—Base Ten \(\rightarrow\) The Number System

Number and Operations—Fractions

K 1 2 3 4 5 6 7 8 High School
## Ties between domains

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<tr>
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<tbody>
<tr>
<td>1</td>
<td>Understand and apply properties of operations and the relationship between addition and subtraction.</td>
<td>Use place value understanding and properties of operations to add and subtract.</td>
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<tr>
<td>2</td>
<td>Use place value understanding and properties of operations to add and subtract.</td>
<td>Use place value understanding and properties of operations to perform multi-digit arithmetic.</td>
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<tr>
<td>3</td>
<td>Understand properties of multiplication and the relationship between multiplication and division.</td>
<td>Use place value understanding and properties of operations to perform multi-digit arithmetic.</td>
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<tr>
<td>4</td>
<td>Use place value understanding and properties of operations to perform multi-digit arithmetic.</td>
<td>Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.</td>
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<tr>
<td>5</td>
<td>Apply and extend previous understandings of multiplication and division to multiply and divide fractions.</td>
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</table>
Fraction clusters, Grades 3–6

Grade 3
- Develop understanding of fractions as numbers.

Grade 4
- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions, and compare decimal fractions.

Grade 5
- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Grade 6
- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Compute fluently with multi-digit numbers and find common factors and multiples.
- Apply and extend previous understandings of numbers to the system of rational numbers.
Sample standard: 4.NF.4

4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

a. Understand a fraction \( \frac{a}{b} \) as a multiple of \( \frac{1}{b} \). For example, use a visual fraction model to represent \( \frac{5}{4} \) as the product \( 5 \times \left( \frac{1}{4} \right) \), recording the conclusion by the equation \( \frac{5}{4} = 5 \times \left( \frac{1}{4} \right) \).

b. Understand a multiple of \( \frac{a}{b} \) as a multiple of \( \frac{1}{b} \), and use this understanding to multiply a fraction by a whole number. For example, use a visual fraction model to express \( 3 \times \left( \frac{2}{5} \right) \) as \( 6 \times \left( \frac{1}{5} \right) \), recognizing this product as \( \frac{6}{5} \). (In general, \( n \times \left( \frac{a}{b} \right) = (n \times a)/b \).)

c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. For example, if each person at a party will eat \( \frac{3}{8} \) of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?
## Functional thinking stream

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<tr>
<td><strong>Patterns</strong></td>
<td>Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.</td>
<td>Generate and analyze patterns. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule “Add 3” and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.</td>
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<tr>
<td><strong>Relationships</strong></td>
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<td>Analyze patterns and relationships. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. For example, given the rule “Add 3” and the starting number 0, and given the rule “Add 6” and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.</td>
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<tr>
<td><strong>Modeling relationships with variables and equations</strong></td>
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<td></td>
<td>Represent and analyze quantitative relationships between dependent and independent variables. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation ( d = 65t ) to represent the relationship between distance and time.</td>
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Look for and make use of structure

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see $7 \times 8$ equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$. 
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Look for and express regularity in repeated reasoning

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation

\[ \frac{y - 2}{x - 1} = 3. \]

Noticing the regularity in the way terms cancel when expanding \((x - 1)(x + 1), (x - 1)(x^2 + x + 1),\) and \((x - 1)(x^3 + x^2 + x + 1)\) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.
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Rain gages

Explain how to make the scale on Gage B. (That is, what function gives the depth \( y \) of the water in the cylinder when \( x \) inches of rain have fallen?) Assume that the radius of the top of the funnel is \( R = 3 \) inches, the radius of the cylinder below it is \( r = 1.5 \) inches, and the height of this cylinder is \( H = 16 \) inches.
(The Abbot of Canterbury’s Puzzle: AD 735–804)

One hundred bushels of corn were distributed among one hundred people in such a way that each man received three bushels, each woman received two bushels, and each child received half a bushel. Given that there were five times as many women as men, how many children were there?
Quantitative reasoning

If everyone in the world went swimming in Lake Michigan, what would happen to the water level? (Would Chicago be flooded?)
Seeing structure in expressions

A physics professor says: “Of course, it is easy to see that

$$L_0 \sqrt{1 - \frac{v^2}{c^2}} = 0$$

when \( v = c \).” Give a possible explanation in terms of the structure of this expression why the professor might say that.
Seeing structure in equations

Do the equations have a solution? Give a reason for your answer that does not depend on solving the equation.

1. \( \frac{t + 2}{3 + t} = 1 \)
2. \( \frac{3 + t}{3 - t} = 1 \)
3. \( \frac{t - 2}{2 - t} = 1 \)
Functional thinking

An American Automobile Association report includes the graph shown below.

*Figure:* Number of drivers involved in fatal crashes per 100,000 population, 1998–2007.
1. Assuming the paper in the roll is very thin, what is the relationship between the thickness of the paper, the inner and outer diameters of the roll, and the length of the paper in the roll? Express your answer as an algebraic formula involving the four listed variables.

2. A roll of masking tape is another example of a tightly rolled spool. In one classroom, layers of masking tape of various thicknesses were measured using a micrometer, a tool for measuring small distances. The table below shows the micrometer readings. . . .
The paper clip below is just over 4 cm long.

How many paper clips like this may be made from a straight piece of wire 10 meters long?