“What Are the Rules?”

How Students’ Mathematical Beliefs and Practices Trap them in a Cycle of Low Achievement

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Some Data

- Nearly half of all college students in the US are currently attending 2-year colleges.

- 70% of those are taking ‘remedial’ math.

- Less than 1 in 10 complete the courses – and college.
Access to algebraic understanding comes from:

- foundational knowledge (eg proportional reasoning)
- ways of thinking and believing
- ways of working learned in earlier years
Community College Students

- Is there a number between 5/6 and 1?
  - A) yes 50%
  - B) no 50%

- Of those saying yes 33% wrote a correct answer
\[ n \times \frac{1}{3} = a \]

a) a is greater than n
b) a is less than n
c) a is equal to n
d) it’s impossible to tell
The Lesson Study Project

- Lesson Study Project was funded by a grant from the Noyce Foundation.

- Lesson Study Team:
  - David Foster, Mathematics Director, Noyce Foundation
  - Cathy Humphreys, middle school Mathematics Coach
  - Jacque Sullivan, middle school SDAIE teacher

- The students we worked with were in a 6th grade SDAIE class at a local middle school.
Inside Jacque Sullivan’s 6th grade SDAIE classroom...

- 29 students: 9 girls, 20 boys

- Proficiency Levels from California Standards Test:
  - 1/3 Far Below Basic
  - 1/2 Below Basic
  - 1/6 Basic

- California English Language Development Test (CELDT):
  - Level 3 40%
  - Level 4 25%
  - Level 5 20%
We decided to focus on how to help the students make sense of fractions as parts of a (geometric) whole.

- Seeing Fractions, California State Department of Education
- Different Shapes, Equal Pieces, TERC
- Mathematical Power, Ruth Parker
- Balanced Assessment, Middle Grades Package 2
Our mathematical goals for students:

- to make sense of fractions geometrically
- to understand area measurement
- to understand fractions as part-to-whole ratios of areas
- to learn to follow mathematical arguments
- to communicate their thinking and justify their solutions
Mathematical Practices in these lessons:

- Making sense of problems and persevering in solving them
- Making sense of quantities and their relationships
- Constructing viable arguments and critiquing the reasoning of others
- Justifying their conclusions
How had the students been accustomed to engaging with mathematics?

- There was one way to do problems.
- Telling them if they were right, or how to do something, provided security.
- They were comfortable with worksheets and writing answers on papers.
- They tended to be “intellectually obedient.”
The lesson sequence emerged as we worked with the students.

- We would figure out what to do next and plan the lessons together, taking turns with teaching.

- When interesting things happened in class, we would, whenever it was feasible, stop the action and have a mini-conference about what to do next (we did this a lot).
Concepts students were developing:

A fraction is the ratio of the area of a part to the area of the whole.

Using congruence to justify equal fractional parts.

Using area to justify equal fractional parts.

Using superpositioning to justify congruence.

Using measurement (square units) to justify equal fractional parts.

Finding shapes with equal areas to justify equal fractional parts.

"If the shape has a line of symmetry, then the two parts are congruent."

"If I can pick up a piece and put it on top of another piece, then the two shapes are congruent."

"If I can divide the shape into unit squares, then I can count the squares to measure the area."

"If I can cut a shape up and put the pieces on top of another shape, then the areas are the same."
“Justify that this square is divided into fourths.”
“What fraction of the whole square is this shaded piece?”
Four different answers:

\[
\frac{1}{8}, \quad \frac{1}{6}, \quad \frac{1}{5}, \quad \frac{1}{4}
\]

Can you figure out the logic behind each of these answers?
“Is this square divided into eighths? Why?”
What do you think Elvis understands about fractions?
“If I shade one part, then $1/8$ is shaded. How do we know?”
“If we were thinking of this as a Crazy Cake, what part of the cake would that person get?”
What do you think now about what Elvis understands about fractions?
Elvis:

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What fraction of the cake is shaded?
I think I am getting $\frac{1}{5}$ of the cake.

Because the question said what fraction of the cake you're getting, it doesn't tell you how much you're getting.

How do you know?
I know that is $\frac{1}{5}$ because you're getting $\frac{1}{5}$ of the cake.

(Hand-drawn diagram of a fraction of a cake)
Alejandro: What fraction of the cake is shaded and why.

I think it is one eighth because the area is 16, so if you divide the go boards into eight equal pieces each one will have two $a^2$. The part that is shaded in has two $a^2$ therfore I think it is one of eight.
“Figure out what fraction each piece is of the whole square; then justify your answer.”

B. There are 36 squares in total.
3 squares are shaded so that makes it $\frac{3}{36} = \frac{1}{12}$.

C. There are 36 squares in total.
5 are shaded so that makes it $\frac{5}{36}$.

D. There are 36 squares in total.
16 are shaded so that makes it $\frac{16}{36} = \frac{8}{18}$.
Eddie Gray & David Tall
(JRME, 1994)

- 72 students 7-13
- ‘above average’ ‘average’ ‘below average’
Addition of a single digit number to a teen digit number with a total below 20, eg 4+13.

Strategies:
- Counting all
- Counting on
- Known facts
- Derived facts – (number sense)
Age 8

‘Above average’:
30% known facts
61% number sense
9% count on

‘Below average’:
6% known facts
0% number sense
72% count on
22% count all
Age 10

- ‘below average’ group used the same number of ‘known facts’ as above average 8 year olds

- Virtually no number sense

- Instead - they count
The difference was not knowing more but:

Flexible thinking
Gray & Tall

- ‘low achievers’ are not slow learners, they are learning a different mathematics

- The mathematics they are learning is ‘a more difficult form of mathematics’
Counting back
Counting grows ever more complex as problems become more difficult.
Hierarchy of concepts

- Counting on
  - Concept of sum
    - Repeated Addition
      - Concept of product
- Counting
  - Concept of Number
Compression
Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through the same process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.

W. T. Thurston
Hierarchy of concepts

- Counting
  - Concept of Number
    - Counting on
      - Concept of sum
        - Repeated Addition
          - Concept of product
“Their persistence in emphasizing procedures leads many children inexorably into a cul-de-sac from which there is little hope of future development.”
They become further apart from the flexible thinkers.
One summer

“You can do things with problems to solve them.” (Jenny)
Low achieving students often:

- Are learning a harder form of mathematics
- Are less likely to view mathematics as a set of numbers or shapes that they can use flexibly
- Compress ideas less
- See mathematics as a set of rules, all equally important
- Are often trapped in “procedural cul-de-sacs”
- Unwilling to act on problems to make them easier
“Obvious is the most dangerous word in mathematics”

Eric Temple-Bell
What fraction of the whole square is shaded?
What is Miguel revealing in his question?
Low achieving students often:

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- See mathematics as a set of rules, all equally important
- Are often trapped in “procedural cul-de-sacs”
- Unwilling to act on problems to make them easier
- Think that each new problem has a new set of rules
- Do not see a role for sense making or reasoning
To conclude

- Low achievers are learning a more difficult form of mathematics
- It will not help to layer more procedures on their ladder of procedures
- They need experience of using mathematics flexibly
- They need to talk
- We don’t need to teach them more as much as we need to change their world views of mathematics, not easy but do-able